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Gradient expansion approach to nonlinear superhorizon perturbations

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Using the gradient expansion approach, we formulate a nonlinear cosmological perturbation theory on super-horizon scales valid to $O(\epsilon^2)$, where ϵ is the expansion parameter associated with a spatial derivative. For simplicity, we focus on the case of a single perfect fluid, but we take into account not only scalar but also vector and tensor modes. We derive the general solution under the uniform-Hubble time-slicing. In doing so, we identify the scalar, vector and tensor degrees of freedom contained in the solution. Then we consider the coordinate transformation to the synchronous gauge to compare our result with the previous result given in the literature.

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I. INTRODUCTION

Recently much attention has been paid to the possibility of non-Gaussian primordial curvature perturbations from inflation. The cosmic microwave background (CMB) temperature anisotropy was found by COBE to be of order 10^{-5} [1], and the recent more accurate observation by WMAP was found to be perfectly consistent with the predictions of the inflationary universe, that the universe is spatially flat and that the primordial curvature perturbation is almost scale-invariant and statistically Gaussian [2, 3]. Thus, it seems that the use of linear cosmological perturbation theory that has been developed in the last couple of decades [4, 5, 6] has been observationally justified.

Nevertheless, the constraint obtained by WMAP on the deviation from Gaussianity is still very weak [3], and the next generation to WMAP, i.e. the PLANCK satellite may be able to detect non-Gaussianity and constrain the level of non-Gaussianity by an order of magnitude better than the WMAP observation with only temperature anisotropies [7], and with combination of temperature and polarization (E mode) anisotropies [8]. Furthermore, it is recently suggested that by observing 21 cm background anisotropies, it might be possible to constrain the primordial non-Gaussianity to the accuracy which is better than the WMAP observation by three orders of magnitude [9]. Also, on the theoretical side, although the non-Gaussianity from the standard single-field slow-roll inflation is too small to be detected [10, 11], many new types of inflationary models have been proposed in the last few years, motivated mainly by string theory or higher dimensional gravity theories, that predict rather large non-Gaussianity [12, 13]. Consequently, studies on the non-Gaussianity from inflation have bloomed [14].

The reason why there can be still relatively large non-Gaussianity is precisely due to the smallness of the primordial curvature perturbation amplitude. This is demonstrated explicitly in terms of the so-called f_{NL} parameter introduced by Komatsu and Spergel [7],

$$\Phi(\boldsymbol{x}) = \Phi_L(\boldsymbol{x}) + f_{NL}\left(\Phi_L^2(\boldsymbol{x}) - \langle\Phi_L^2\rangle\right), \tag{1.1}$$

where Φ is the curvature perturbation on the Newton (or longitudinal) slicing and Φ_L is the linear Gaussian part of the perturbation. Since $\Phi_L \sim 10^{-5}$, the coefficient f_{NL} can be very large even if Φ is still to be very small.

Thus, to quantify the non-Gaussianity and clarify its observational effect, it is important to develop a theory that can deal with nonlinear cosmological perturbations. There are a couple of approaches to nonlinear perturbations. One is a second-order perturbation theory, which has been used by several authors to quantify the non-Gaussianity [10, 11, 12, 15]. Another is to invoke gradient expansion by assuming the spatial derivative is sufficiently small compared to the time derivative [16]. In this paper we take the gradient expansion approach and formulate the cosmological perturbation to full nonlinear order in its amplitude but to second order in spatial gradients.

In passing we note that while we work with the standard metric perturbation, there exists a different approach, called the covariant approach [17]. In particular, recently Langlois and Vernizzi have succeeded to formulate it in the form that may be useful when dealing with nonlinear perturbations not only on superhorizon scales but also on subhorizon scales [18].

In the (spatial) gradient expansion approach, we introduce an expansion parameter, ϵ, and associate it with each spatial derivative, expand the field equations in terms of ϵ, and solve them order by order iteratively. Historically, gradient expansion was used to explore the general behavior of the spacetime near the cosmological singularity [19, 20]. Tomita called it the anti-Newtonian approximation [21] and used it to investigate the cosmological perturbations on superhorizon scales [22]. The terminology, 'gradient expansion' is perhaps due to Salopek and Bond [23]. They adapted it to the Hamilton-Jacobi formalism and used it to investigate the nonlinear perturbations on superhorizon scales in slow-roll inflation. Then Comer et al.[24] formulated gradient expansion and discussed the form of the solution to $O(\epsilon^4)$ beyond the leading order, and Deruelle and Langlois made a thorough study on nonlinear perturbations near the cosmological singularity [25].

The perturbations on superhorizon scales has been studied extensively in the so-called separate universe approach [26, 27]. The fact that the separate universe approach is essentially the leading order approximation to gradient expansion was demonstrated by Rigopoulos and Shellard [28], and a more general analysis was given by Lyth, Malik and Sasaki [16].

The most of the previous studies were, however, confined to the leading order approximation to gradient expansion, including the apparent case of the separate universe approach. Although this leading order approximation seems to be justifiable for quite a large number of models, it can miss important points in some cases. A good example is the case studied by Leach et al. [29]. They considered the linear perturbation of a single-field inflation model which has a stage at which the slow-roll conditions are violated. It is then shown that the $O(k^2)$ corrections to the curvature perturbation on superhorizon scales, where k is the comoving wavenumber of the perturbation, plays a crucial role in the determination of the final curvature perturbation amplitude. To be a bit more precise, it is the decaying mode solution which becomes non-negligible and makes the $O(k^2)$ corrections important. If the $O(k^2)$ corrections are crucial already at linear order, one expects that they will affect the non-Gaussianity significantly. In gradient expansion, they correspond to $O(\epsilon^2)$ beyond the leading order. Thus it is important to include the $O(\epsilon^2)$ terms in gradient expansion.

In the above context, i.e., considering the application of gradient expansion to the inflationary cosmology, the most relevant formulation that includes the $O(\epsilon^2)$ corrections seems to be the one by Comer et al. [24]. However, they worked in the synchronous gauge. Of course nothing is particularly wrong with the synchronous gauge, but it is not the most convenient choice for the analysis of superhorizon scale perturbations. One apparent drawback is the existence of a gauge mode. In other words, the synchronous time-slicing condition does not fix the slices uniquely. Furthermore, they assumed a certain, special form of the metric, and did not give solutions of all the variables but the metric. This obscured the physical degrees of freedom associated with their solution.

While nothing remains unclear about the gauge-dependence and/or gauge-invariance in the linear perturbation theory, there are still some issues to be clarified in the case of nonlinear, superhorizon perturbations. Among them are the issue of identifying the scalar, vector and tensor modes and their gauge (or coordinate) transformation properties. In the separate universe approach, or equivalently at leading order of gradient expansion, the uniform density slicing (or equivalently uniform Hubble slicing) turns out to be convenient [26, 30], and a nonlinear version of a gauge-invariant combination that corresponds to the curvature perturbation on the uniform density slicing has been constructed [16]. This quantity is independent of the choice of slicing and conserved for adiabatic perturbations, thus represents a nonlinear scalar mode perturbation. However, tensor modes which are to describe gravitational waves are not clearly seen at leading order. This is because of the general covariance that states that even if there exits a gravitational wave, spacetime can be made locally Minkowski within a region much smaller than the wavelength of the gravitational wave. Thus it seems necessary to go to higher orders to make clear distinction, if any, of tensor, scalar and vector modes.

As a first step toward understanding these issues, in this paper we formulate the gradient expansion to $O(\epsilon^2)$ beyond leading order in the uniform Hubble slicing. We employ the $(3+1)$-decomposition (Hamiltonian formalism) of the Einstein equations. For simplicity, we assume a perfect fluid equation of state $P = (\Gamma - 1)\rho$ with Γ =const.. We then derive the general solution for all the variables, focusing particularly on the correspondence between the degrees of freedom appearing in the general solution and those in the linear theory. We find that the identification of the tensor mode in the spatial metric is rather arbitrary, depending on how one fixes the spatial coordinates, as a reflection of the general covariance, while it can be unambiguously identified in the extrinsic curvature of the metric, albeit non-locally.

Next we consider a coordinate transformation from the uniform Hubble slicing to the synchronous slicing to compare our result with that of Comer et al. [24]. The purpose is to clarify the relation between the variables defined in each coordinate system. But this is also a good check on the computation. We find a perfect agreement.

This paper is organized as follows. In Section II, we define the basic variables we use in our paper using the $(3+1)$ decomposition. In Section III, we first describe the basic assumptions for gradient expansion. Then we derive the general solution on uniform Hubble slicing to $O(\epsilon^2)$ beyond leading order, and identify the scalar, vector and tensor modes in the general solution. In Section IV, we consider the transformation from the uniform Hubble slicing to the synchronous slicing, and clarify the relation of the result obtained by Comer et al. [24] with our solution. We conclude our paper in Section V. Some useful order-counting formulas are given in Appendix.

II. BASIC EQUATIONS

In the $(3+1)$-decomposition, the metric is expressed as

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= (-\alpha^2+\beta_k\beta^k)dt^2+2\beta_i dx^i dt+\gamma_{ij}dx^i dx^j, \end{aligned} \tag{2.1}$$

where α, β^i ($\beta_i=\gamma^{ij}\beta^j$), and γ_{ij} are the lapse function, shift vector, and the 3-dimensional spatial metric, respectively. We rewrite γ_{ij} as

$$\gamma_{ij}(t,x^k)=a^2(t)\,\psi^4(t,x^k)\,\tilde\gamma_{ij}(t,x^k)\,;\quad \det(\tilde\gamma_{ij})=1\,, \tag{2.2}$$

where the function $a(t)$ is the scale factor of a fiducial homogeneous and isotropic background universe.

The extrinsic curvature K_{ij} of the $t=$const. hypersurface is defined by

$$K_{ij}\equiv-\nabla_i n_j=\alpha\Gamma^0{}_{ij}\,, \tag{2.3}$$

where $n_\mu=(-\alpha,0,0,0)$ is the vector unit normal to the time slices. We decompose the extrinsic curvature as

$$K_{ij}=\frac{\gamma_{ij}}{3}K+\psi^4a^2\tilde A_{ij}\,;\quad K\equiv\gamma^{ij}K_{ij}\,, \tag{2.4}$$

where $\tilde A_{ij}$ represents the traceless part of K_{ij} The factors in front of $\tilde A_{ij}$ are determined so that the mixed components $K^i{}_j$ takes the form,

$$K^i{}_j=\frac{1}{3}\delta^i_jK+\tilde A^i{}_j\,, \tag{2.5}$$

where the indices of $\tilde A_{ij}$ are to be raised or lowered by $\tilde\gamma^{ij}$ and $\tilde\gamma_{ij}$.

The stress-energy tensor for the fluid is

$$T_{\mu\nu}=(\rho+P)u_\mu u_\nu+Pg_{\mu\nu}, \tag{2.6}$$

where ρ, P and u^μ are the energy density, pressure, and the 4-velocity, respectively. We consider a perfect fluid with the barotropic equation of state $P/\rho=\Gamma-1=$const.. The components of the 4-velocity are expressed as

$$\begin{aligned} &u^0=[\alpha^2-(\beta_k+v_k)(\beta^k+v^k)]^{-1/2},\quad u^i=u^0v^i,\quad v^i\equiv\gamma^{ij}v_j,\\ &u_0=-u^0[\alpha^2-\beta^k(\beta_k+v_k)],\quad u_i=u^0(v_i+\beta_i). \end{aligned} \tag{2.7}$$

We express the $(3+1)$-decomposition of the energy-momentum tensor as

$$E\equiv T_{\mu\nu}n^\mu n^\nu=(\rho+P)(\alpha u^0)^2-P, \tag{2.8}$$

$$J_j\equiv-T_{\mu\nu}n^\mu\gamma^\nu_j=(\rho+P)\alpha u^0u_j, \tag{2.9}$$

$$S_{ij}\equiv T_{ij}=(\rho+P)(u^0)^2(v_i+\beta_i)(v_j+\beta_j)+P\gamma_{ij}, \tag{2.10}$$

$$S^k{}_k\equiv\gamma^{kl}S_{lk}. \tag{2.11}$$

The hydrodynamic equations $\nabla^\mu T_{\mu\nu}=0$ are written in the form [30]

$$\partial_t(w\psi^6a^3\rho^{1/\Gamma})+\partial_k(w\psi^6a^3\rho^{1/\Gamma}v^k)=0, \tag{2.12}$$

$$\begin{aligned} &\partial_t\{w\psi^6a^3(\rho+P)u_j\}+\partial_k\{w\psi^6a^3(\rho+P)v^ku_j\}\\ &\quad=-\alpha\psi^6a^3\partial_jP+w\psi^6a^3(\rho+P)\{-\alpha u^0\partial_j\alpha+u_k\partial_j\beta^k-\frac{u_ku_l}{2u^0}\partial_j\gamma^{kl}\}, \end{aligned} \tag{2.13}$$

where $w\equiv\alpha u^0$, and

$$v^k\equiv\frac{u^k}{u^0}=-\beta^k+\tilde\gamma^{kl}\frac{u_l}{\psi^4a^2u^0}. \tag{2.14}$$

In the $(3+1)$-formalism of the Einstein equations, the dynamical variables are γ_{ij} and K_{ij}. The (n,n) and (n,j) components of the Einstein equations give the Hamiltonian and momentum constraint equations, respectively, while the (i,j) components gives the evolution equations for K_{ij}. The evolution equations for γ_{ij} are given by the definitions of the extrinsic curvature (2.3).

In the present case, the Hamiltonian and momentum constraints are

$$R - \tilde{A}_{ij}\tilde{A}^{ij} + \frac{2}{3}K^2 = 16\pi G E\,, \tag{2.15}$$

$$D_i \tilde{A}^i{}_j - \frac{2}{3}D_j K = 8\pi G J_j\,. \tag{2.16}$$

The evolution equations for γ_{ij} are given as

$$(\partial_t - \beta^k\partial_k)\psi + \frac{\dot{a}}{2a}\psi = \frac{\psi}{6}\{-\alpha K + \partial_k\beta^k\}, \tag{2.17}$$

$$(\partial_t - \beta^k\partial_k)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta^k + \tilde{\gamma}_{jk}\partial_i\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k\,, \tag{2.18}$$

where $\dot{} = d/dt$. The evolution equations for K_{ij} are given as

$$\begin{aligned}(\partial_t - \beta^k\partial_k)K &= \alpha(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2) - D_k D^k\alpha \\ &\quad +4\pi G\alpha(E + S^k{}_k),\end{aligned} \tag{2.19}$$

$$\begin{aligned}(\partial_t - \beta^k\partial_k)\tilde{A}_{ij} &= \frac{1}{a^2\psi^4}[\alpha(R_{ij} - \frac{\gamma_{ij}}{3}R) \\ &\quad -(D_iD_j\alpha - \frac{\gamma_{ij}}{3}D_kD^k\alpha)] + \alpha(K\tilde{A}_{ij} - \tilde{A}_{ik}\tilde{A}^k{}_j) \\ &\quad +\tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{jk}\partial_i\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k - \frac{8\pi G\alpha}{a^2\psi^4}(S_{ij} - \frac{\gamma_{ij}}{3}S^k{}_k),\end{aligned} \tag{2.20}$$

where R_{ij} is the Ricci tensor of the metric γ_{ij}, $R \equiv \gamma^{ij}R_{ij}$, and D_i is the covariant derivative with respect to γ_{ij}.

We define the local Hubble parameter as $1/3$ of the expansion of the unit normal vector n^μ, which is equal to $-1/3$ of the trace of the extrinsic curvature in our convention,

$$3H \equiv -K = \frac{3\dot{a}}{\alpha a} + 6\frac{\partial_t\psi}{\psi} - \frac{D_i\beta^i}{\alpha}\,. \tag{2.21}$$

In the following, we adopt the uniform Hubble slicing. For this slicing, we have

$$H(t) = \frac{\dot{a}}{a}\,, \tag{2.22}$$

and Eq. (2.21) implies

$$\alpha = \frac{1 - \dfrac{D_i\beta^i}{3H}}{1 - \dfrac{2\partial_t\psi}{H\psi}}\,. \tag{2.23}$$

III. GRADIENT EXPANSION

A. Assumptions

We consider nonlinear superhorizon perturbations in the approach called the anti-Newtonian approximation [22], the spatial gradient expansion [23], or the long wavelength approximation [30]. In this approach, we assume that the characteristic length scale L of inhomogeneities is always much larger than the Hubble horizon scale, $L \gg H^{-1} \sim t$. We introduce a small parameter ϵ, and assume that L is of $O(1/\epsilon)$.

This assumption is equivalent to assuming that the magnitude of spatial gradients of the quantities is given by $\partial_i\psi = \psi \times O(\epsilon)$, $\partial_i\alpha = \alpha \times O(\epsilon)$, and so on. In the limit $L \to \infty$, i.e., $\epsilon \to 0$, the universe looks locally like a FLRW spacetime, where 'locally' means as seen on the scale of the Hubble horizon volume. This implies that anisotropic quantities, i.e., β^i and v^i should be at least $O(\epsilon)$. Also, for the spatial metric to be locally homogeneous and isotropic, $\tilde{\gamma}_{ij}$ should be time independent in the limit $\epsilon \to 0$ [16]. This implies $\partial_t \tilde{\gamma}_{ij} = O(\epsilon)$. However, as discussed in Section III.C, allowing it to be of $O(\epsilon)$ makes the analysis too complicated. We therefore require a bit tighter condition, $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$. Thus our basic assumptions are

$$\beta^i = O(\epsilon)\,, \quad v^i = O(\epsilon)\,, \quad \partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)\,. \tag{3.1}$$

We note that from Eq. (2.18), the assumptions $\beta^i = O(\epsilon)$ and $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$ implies $\tilde{A}_{ij} = O(\epsilon^2)$.

It is noted that physical quantities which are approximately homogeneous on each Hubble horizon scale can vary nonlinearly on very large scales. Thus this approach at leading order is called the separate universe hypothesis [26, 27].

We introduce

$$\delta \equiv \rho/\rho_0 - 1, \tag{3.2}$$

where ρ_0 is a fiducial homogeneous part of ρ, which is defined by

$$\rho_0(t) = \frac{3H^2(t)}{8\pi G}\,, \tag{3.3}$$

where $H(t)$ is given by Eq. (2.22). Then, using the equations presented in the previous section we can evaluate the order of magnitude of the basic variables as [30]

$$\begin{aligned} &\psi = O(1)\,, \quad \chi \equiv \alpha - 1 = O(\epsilon^2)\,, \\ &\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)\,, \quad \partial_t \psi = O(\epsilon^2)\,, \quad \tilde{A}_{ij} = O(\epsilon^2)\,, \\ &\delta = O(\epsilon^2)\,, \quad u_i = O(\epsilon^3)\,, \quad \beta^i + v^i = O(\epsilon^3)\,. \end{aligned} \tag{3.4}$$

An explicit demonstration of the above evaluation is given in Appendix.

In particular, the fact that ψ is constant in time to $O(\epsilon)$ follows from the perfect fluid assumption $P = P(\rho)$. This corresponds to the conservation of the adiabatic curvature perturbation in linear theory. Hence we may regard ψ as a nonlinear generalization of the curvature perturbation, although the actual spatial curvature R is no longer described solely in terms of ψ in the nonlinear case.

We also note that the curvature perturbation is defined in several different ways in the literature. The correspondence of ψ to the conventions used in a couple of other papers are

$$\psi^2 = \exp[\,\psi_{\rm LMS}\,] = \exp[\,\zeta_{\rm Mal}\,]\,, \tag{3.5}$$

where $\psi_{\rm LMS}$ and $\zeta_{\rm Mal}$ are used by Lyth et al. [16] and Maldacena [11], respectively.

B. The leading order solutions

First we derive the leading order solutions for our basic variables. We note that this does not mean we only solve the Einstein equations at lowest order in gradient expansion. For example, from Eq. (3.4), we have $\tilde{A}_{ij} = O(\epsilon^2)$. Hence we may neglect it at lowest order in gradient expansion, while here we do solve for the leading order part of $\tilde{A}_{ij}$.

Taking $O(\epsilon^0)$ part of Eqs. (2.15) and (2.16), we have

$$\frac{2}{3}K^2 = 16\pi G\rho_0, \tag{3.6}$$

$$\partial_t K = \frac{1}{3}K^2 + 4\pi G(3\Gamma - 2)\rho_0. \tag{3.7}$$

Integrating these equations, we obtain

$$a(t) = a_f t^{\frac{2}{3\Gamma}}\,, \tag{3.8}$$

$$\rho_0(t) = \frac{1}{6\pi G\Gamma^2 t^2}, \tag{3.9}$$

where a_f is a constant whose normalization is arbitrary.

Substituting the order of magnitude evaluation of the variables shown in Eq. (3.4) into the hydrodynamic equations given in Sec. II, we find

$$\frac{1}{\Gamma}\partial_t\delta + \frac{6\partial_t\psi}{\psi} + D_k v^k = O(\epsilon^4), \tag{3.10}$$

$$\partial_t\{\psi^6\rho_0 a^3\Gamma u_j\} = -\psi^6\rho_0 a^3[\Gamma\partial_j\chi + (\Gamma-1)\partial_j\delta] + O(\epsilon^5)\,. \tag{3.11}$$

The Hamiltonian and momentum constraint equations give

$$\tilde{\gamma}^{ij}\tilde{D}_i\tilde{D}_j\psi = \frac{1}{8}\tilde{\gamma}^{kl}\tilde{R}_{kl}\psi - 2\pi G\psi^5 a^2\rho_0\delta + O(\epsilon^4), \tag{3.12}$$

$$\tilde{D}^j(\psi^6\tilde{A}_{ij}) = 8\pi G\Gamma\rho_0 u_i\psi^6 + O(\epsilon^5), \tag{3.13}$$

where $\tilde{R}_{ij}$ is the Ricci tensor with respect to $\tilde{\gamma}_{ij}$, $\tilde{D}_i$ is the covariant derivative with respect to $\tilde{\gamma}_{ij}$.

The evolution equations for the spatial metric give

$$\frac{6\partial_t\psi}{\psi} - 3\frac{\dot{a}}{a}\chi = D_k\beta^k, \tag{3.14}$$

$$(\partial_t - \beta^k\partial_k)\tilde{\gamma}_{ij} = -2\tilde{A}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta^k + \tilde{\gamma}_{jk}\partial_i\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k + O(\epsilon^4)\,, \tag{3.15}$$

while the evolution equations for the extrinsic curvature give

$$\partial_t\tilde{A}_{ij} + 3\frac{\dot{a}}{a}\tilde{A}_{ij} = \frac{1}{a^2\psi^4}[R_{ij} - \frac{\gamma_{ij}}{3}R] + O(\epsilon^4)\,, \tag{3.16}$$

$$\nabla^2\chi = 4\pi G\rho_0 a^2\{(3\Gamma-2)\delta + 3\Gamma\chi\} + O(\epsilon^4)\,, \tag{3.17}$$

where

$$\nabla^2 \equiv \frac{1}{\psi^6}\partial_k(\psi^2\tilde{\gamma}^{kl}\partial_l)\,. \tag{3.18}$$

We now solve the above equations to derive the leading order general solution of each variable. From Eq. (3.4), we have

$$\psi = {}_{(0)}L(x^i) + O(\epsilon^2), \tag{3.19}$$

where ${}_{(0)}L$ is an arbitrary function of the spatial coordinates and of $O(\epsilon^0)$. Here and in what follows, the prefix (n) means the quantity is of $O(\epsilon^n)$.

From Eq. (3.10) and Eq. (3.14), we obtain

$$\partial_t\delta + 3\Gamma\frac{\dot{a}}{a}\chi = O(\epsilon^4). \tag{3.20}$$

Since the left-hand side of Eq. (3.17) is of $O(\epsilon^4)$, the fact that both δ and χ are $O(\epsilon^2)$ gives

$$\chi = -\frac{3\Gamma-2}{3\Gamma}\,\delta + O(\epsilon^4). \tag{3.21}$$

From Eq. (3.20) and Eq. (3.21), we have

$$\partial_t\delta - (3\Gamma-2)\frac{\dot{a}}{a}\delta = O(\epsilon^4). \tag{3.22}$$

This equation is integrated to give

$$\delta = {}_{(2)}Q(x^i)t^{2-\frac{4}{3\Gamma}} + O(\epsilon^4)\,. \tag{3.23}$$

Then Eq. (3.21) gives

$$\chi = -\frac{3\Gamma-2}{3\Gamma}\,{}_{(2)}Q(x^i)\,t^{2-\frac{4}{3\Gamma}} + O(\epsilon^4)\,. \tag{3.24}$$

Next, we seek for the general solution of u_l. Using Eq. (3.21), Eq. (3.11) becomes

$$\partial_t\{\rho_0 a^3 \Gamma u_j\} = \frac{\rho_0 a^3 \partial_j \delta}{3} + O(\epsilon^5). \tag{3.25}$$

Solving this equation, we have obtain

$$u_j = \frac{\partial_{j\,(2)}Q(x^i)}{3\Gamma+2} t^{3-\frac{4}{3\Gamma}} + {}_{(3)}C_j(x^i) t^{2-\frac{2}{\Gamma}}, \tag{3.26}$$

where ${}_{(3)}C_j(x^i)$ is an arbitrary vector of the spatial coordinates.

To find the leading order solution of $\tilde{A}_{ij}$, we need to evaluate the spatial Ricci tensor R_{ij}. For this purpose, we first consider the form of the spatial metric. Since $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$, we define

$$H_{ij}(t,x^i) \equiv \tilde{\gamma}_{ij} - {}_{(0)}f_{ij}(x^i) = O(\epsilon^2), \tag{3.27}$$

where ${}_{(0)}f_{ij}$ is an arbitrary function of the spatial coordinates. Namely, $H_{ij}(t,x^i)$ is the $O(\epsilon^2)$ part of $\tilde{\gamma}_{ij}$. Then the Ricci tensor of the spatial metric $\psi^4 \tilde{\gamma}_{ij}$ can be decomposed as

$$R_{ij} = \tilde{R}_{ij} + R^{\psi}_{ij}\,, \tag{3.28}$$

where $\tilde{R}_{ij}$ is the Ricci tensor with respect to $\tilde{\gamma}_{ij}$, and

$$R^{\psi}_{ij} \equiv -\frac{2}{\psi}\tilde{D}_i\tilde{D}_j\psi - \frac{2}{\psi}\tilde{\gamma}_{ij}\tilde{\Delta}\psi + \frac{6}{\psi^2}\tilde{D}_i\psi\tilde{D}_j\psi - \frac{2}{\psi^2}\tilde{\gamma}_{ij}\tilde{D}_k\psi\tilde{D}^k\psi\,, \tag{3.29}$$

where $\tilde{\Delta}$ is the Laplacian with respect to $\tilde{\gamma}_{ij}$. Using Eqs. (3.27) and (3.19), we have

$$\begin{aligned} \tilde{R}_{ij} &= {}_{(2)}\bar{R}_{ij} + O(\epsilon^4), \\ R^{\psi}_{ij} &= {}_{(2)}R^L_{ij} + O(\epsilon^4), \end{aligned} \tag{3.30}$$

where ${}_{(2)}\bar{R}_{ij}$ is the Ricci tensor with respect to ${}_{(0)}f_{ij}$, and

$${}_{(2)}R^L_{ij} \equiv -\frac{2}{{}_{(0)}L}\bar{D}_i\bar{D}_{j\,(0)}L - \frac{2}{{}_{(0)}L}{}_{(0)}f_{ij}\bar{\Delta}_{(0)}L + \frac{6}{{}_{(0)}L^2}\bar{D}_{i\,(0)}L\bar{D}_{j\,(0)}L - \frac{2}{{}_{(0)}L^2}{}_{(0)}f_{ij}\bar{D}_{k\,(0)}L\bar{D}^k{}_{(0)}L, \tag{3.31}$$

where $\bar{D}_i$ is the covariant derivative with respect to ${}_{(0)}f_{ij}$, and $\bar{\Delta}$ is the Laplacian with respect to ${}_{(0)}f_{ij}$.

Now, we seek for the general solution of $\tilde{A}_{ij}$. Using Eqs. (3.28) and (3.30), the evolution equation for $\tilde{A}_{ij}$, Eq. (3.16), becomes

$$\partial_t\tilde{A}_{ij} + 3\frac{\dot{a}}{a}\tilde{A}_{ij} = \frac{{}_{(2)}F_{ij}(x^k)}{a^2} + O(\epsilon^4), \tag{3.32}$$

where

$${}_{(2)}F_{ij} \equiv \frac{1}{{}_{(0)}L^4}\left[{}_{(2)}\bar{R}_{ij} + {}_{(2)}R^L_{ij} - \frac{{}_{(0)}f_{ij}}{3}\,{}_{(0)}f^{kl}\left({}_{(2)}\bar{R}_{kl} + {}_{(2)}R^L_{kl}\right)\right]. \tag{3.33}$$

Eq. (3.32) is immediately integrated to give

$$\tilde{A}_{ij} = \frac{{}_{(2)}F_{ij}}{a^3}\int^t a(t')dt' + \frac{{}_{(2)}C_{ij}(x^k)}{a^3} + O(\epsilon^4), \tag{3.34}$$

where ${}_{(2)}C_{ij}$ is an arbitrary symmetric, traceless tensor of the spatial coordinates. Thus we obtain

$$\tilde{A}_{ij} = \frac{3\Gamma}{a_f^2(3\Gamma+2)}\,{}_{(2)}F_{ij}(x^k)\,t^{1-\frac{4}{3\Gamma}} + \frac{{}_{(2)}C_{ij}(x^k)}{a_f^3}t^{-\frac{2}{\Gamma}}. \tag{3.35}$$

Here we note that, since the last term proportional to ${}_{(2)}C_{ij}\,a^{-3}$ is a homogeneous solution to Eq. (3.32), it may be of $O(\epsilon)$ in principle. However, allowing it to be of $O(\epsilon)$ alters the $O(\epsilon^2)$ part of the Hamiltonian constraint (3.12) substantially because the quadratic term $\tilde{A}_{ij}\tilde{A}^{ij}$ will no longer be negligible. This is essentially the same as the case of homogeneous but anisotropic cosmologies [20, 22, 25], in which the square of the shear (the traceless part of the extrinsic curvature)works as an effective energy density proportional to a^{-6}. To avoid this complication, for the sake of simplicity, we have assumed $\partial_t\tilde{\gamma}_{ij} = O(\epsilon^2)$, which implies $\tilde{A}_{ij} = O(\epsilon^2)$. We have not investigated how restrictive this additional assumption is. However, in the case of an accelerating universe, $0 < \Gamma < 2/3$, which would mimic an inflationary universe, one expects this decaying mode ($\propto a^{-3}$) to be of order $H(k/(aH))^N$ where $N = 3(2-\Gamma)/(2-3\Gamma)$ so that $H(k/(aH))^N \propto a^{-3}$. Since $N > 3$ for $0 < \Gamma < 2/3$, its order will be higher than $O(\epsilon^3)$. Thus the assumption $\tilde{A}_{ij} = O(\epsilon^2)$ will be safely satisfied.

Up to now we have not considered the Hamiltonian and momentum constraint equations. The constraint equations will relate the arbitrary spatial quantities ${}_{(0)}L$, ${}_{(0)}f_{ij}$, ${}_{(2)}Q$, ${}_{(2)}C_{ij}$ and ${}_{(3)}C_j$.

The $O(\epsilon^2)$ part of Hamiltonian constraint (3.12) yields

$$\begin{aligned} {}_{(2)}Q &= \frac{3\Gamma^2}{a_f^2\,{}_{(0)}L^5}\left[-{}_{(0)}f^{ij}\bar{D}_i\bar{D}_j\,{}_{(0)}L + \frac{1}{8}{}_{(0)}f^{kl}\,{}_{(2)}\bar{R}_{kl}\,{}_{(0)}L\right] + O(\epsilon^4), \\ &= \frac{3\Gamma^2}{8a_f^2\,{}_{(0)}L^4}\,{}_{(0)}f^{kl}\left[{}_{(2)}R^L_{kl} + {}_{(2)}\bar{R}_{kl}\right] + O(\epsilon^4) = \frac{3\Gamma^2}{8a_f^2}\,R\left[L^4 f\right] + O(\epsilon^4)\,, \end{aligned} \tag{3.36}$$

where $R\left[L^4 f\right]$ is the Ricci scalar of the metric ${}_{(0)}L^4\,{}_{(0)}f_{ij}$. Thus ${}_{(2)}Q$ is not arbitrary, but is expressed in terms of ${}_{(0)}f_{ij}$ and ${}_{(0)}L$.

The $O(\epsilon^3)$ part of the momentum constraint (3.13) yields

$$\bar{D}^j\left[{}_{(0)}L^6\tilde{A}_{ij}\right] = 8\pi G\Gamma\rho_0 u_i\,{}_{(0)}L^6 + O(\epsilon^5). \tag{3.37}$$

Inserting Eqs. (3.35) and (3.26) into this equation, we find

$$\frac{\partial_i\,{}_{(2)}Q}{3\Gamma+2} = \frac{9\Gamma^2}{4a_f^2(3\Gamma+2)\,{}_{(0)}L^6}\,\bar{D}^j\left[{}_{(0)}L^6\,{}_{(2)}F_{ij}\right], \tag{3.38}$$

$${}_{(3)}C_i = \frac{3\Gamma}{4a_f^3\,{}_{(0)}L^6}\,\bar{D}^j\left[{}_{(0)}L^6\,{}_{(2)}C_{ij}\right]. \tag{3.39}$$

The latter equation implies ${}_{(3)}C_j$ is not arbitrary but expressed in terms of ${}_{(0)}L$, ${}_{(0)}f_{ij}$ and ${}_{(2)}C_{ij}$, while the former equation is found to be consistent with Eq. (3.36). This consistency is a result of the Bianchi identities.

Before closing this subsection, we mention the following fact. We have considered u_j in the above but not considered the 3-velocity v^i and the shift vector β^i individually. As seen from Eq. (2.14), we have $v^i + \beta^i = u^0\gamma^{ij}u_j$. Thus once we fix the shift vector under the assumption $\beta^i = O(\epsilon)$, v^i is immediately determined. The reason why we have not bothered to do this is that it was unnecessary to fix the shift vector for the derivation of the all the other quantities as far as their leading order terms are concerned. Geometrically u_j does not depend on the choice of the shift vector because it describes the components projected on the t =constant hypersurface. And the same is true for all the other variables except for v^i. This shift vector independence is probably a result of the localness of gradient expansion.

C. The solution to $O(\epsilon^2)$ in gradient expansion

Now we consider the general solution valid to $O(\epsilon^2)$ in gradient expansion. As we have seen in the previous subsection, among the basic variables we have introduced, the only quantities whose leading order terms are lower than $O(\epsilon^2)$ are ψ and $\tilde{\gamma}_{ij}$, that is, the spatial metric. Hence what we have to do is to evaluate the next order terms of these variables.

To find the next order solutions for ψ and $\tilde{\gamma}_{ij}$, we now have to specify the shift vector. For simplicity, we choose β^i to be of $O(\epsilon^3)$. We note that this choice includes the case of the comoving coordinates $v^i = 0$, as seen from Eq. (3.4).

Under the assumption $\beta^i = O(\epsilon^3)$, we can easily find their solutions. From Eq. (3.14), we have

$$\frac{6\partial_t\psi}{\psi} - 3\frac{\dot{a}}{a}\chi = O(\epsilon^4). \tag{3.40}$$

Substituting the solution (3.24) for χ, this equation readily integrated to give

$$\psi = {}_{(0)}L - \frac{{}_{(0)}L\,{}_{(2)}Q}{6\Gamma}t^{2-\frac{4}{3\Gamma}} + {}_{(2)}L(x^i) + O(\epsilon^4), \tag{3.41}$$

where ${}_{(2)}L$ is an arbitrary function of the spatial coordinates. Without loss of generality, we may absorb ${}_{(2)}L$ into ${}_{(0)}L$. Thus we obtain the final expression for ψ as

$$\psi = {}_{(0)}L(x^i) - \frac{{}_{(0)}L(x^i)\,{}_{(2)}Q(x^i)}{6\Gamma}\, t^{2-\frac{4}{3\Gamma}} + O(\epsilon^4). \tag{3.42}$$

Turning to $\tilde{\gamma}_{ij}$, from Eq. (3.15), we have

$$\partial_t \tilde{\gamma}_{ij} = -2\tilde{A}_{ij} + O(\epsilon^4). \tag{3.43}$$

Substituting the solution (3.34) for $\tilde{A}_{ij}$, this equation gives

$$H_{ij} = -2{}_{(2)}F_{ij} \int^t \frac{dt}{a^3(t)} \int^t a(t)dt - 2{}_{(2)}C_{ij} \int^t \frac{dt}{a^3(t)} + O(\epsilon^4), \tag{3.44}$$

The integrals are easily done to yield

$$H_{ij} = -\frac{{}_{(2)}F_{ij}}{a_f^2} \frac{9\Gamma^2}{9\Gamma^2 - 4} t^{2-\frac{4}{3\Gamma}} - \frac{2\Gamma\,{}_{(2)}C_{ij}}{a_f^2(\Gamma - 2)}\, t^{1-\frac{2}{\Gamma}} + {}_{(2)}f_{ij}(x^k) + O(\epsilon^4), \tag{3.45}$$

where ${}_{(2)}f_{ij}$ is an arbitrary function of the spatial coordinates. The final expression for $\tilde{\gamma}_{ij}$ is given by

$$\begin{aligned}
\tilde{\gamma}_{ij} &= {}_{(0)}f_{ij}(x^k) - \frac{{}_{(2)}F_{ij}(x^k)}{a_f^2} \frac{9\Gamma^2}{9\Gamma^2 - 4} t^{2-\frac{4}{3\Gamma}} - \frac{2\Gamma\,{}_{(2)}C_{ij}(x^k)}{a_f^2(\Gamma - 2)}\, t^{1-\frac{2}{\Gamma}} + O(\epsilon^4) \\
&= {}_{(0)}f_{ik} \left(\delta^k_j - {}_{(2)}F^k{}_j \frac{9\Gamma^2}{a_f^2(9\Gamma^2 - 4)}\, t^{2-\frac{4}{3\Gamma}} - {}_{(2)}C^k{}_j \frac{2\Gamma}{a_f^2(\Gamma - 2)}\, t^{1-\frac{2}{\Gamma}} \right) + O(\epsilon^4)\,,
\end{aligned} \tag{3.46}$$

where we have absorbed ${}_{(2)}f_{ij}$ into ${}_{(0)}f_{ij}$. Here, it is important to note that ${}_{(0)}f_{ij}$ is not completely arbitrary but its determinant must be unity, $\det({}_{(0)}f_{ij}) = 1 + O(\epsilon^4)$.

D. Summary of the general solution

We now have the general solutions valid up through $O(\epsilon^2)$ for all the physical quantities on uniform Hubble slicing, under the condition $\beta^i = O(\epsilon^3)$. Let us summarize them.

$$\alpha = 1 + \chi = 1 - {}_{(2)}Q\, \frac{3\Gamma - 2}{3\Gamma}\, t^{2-\frac{4}{3\Gamma}}\,, \tag{3.47}$$

$$\psi = {}_{(0)}L \left(1 - {}_{(2)}Q\, \frac{1}{6\Gamma} t^{2-\frac{4}{3\Gamma}} \right) = {}_{(0)}L \left(1 - \frac{\Gamma}{16 a_f^2} R\left[L^4 f\right]\, t^{2-\frac{4}{3\Gamma}} \right)\,, \tag{3.48}$$

$$\tilde{\gamma}_{ij} = {}_{(0)}f_{ik} \left(\delta^k_j - {}_{(2)}F^k{}_j \frac{9\Gamma^2}{a_f^2(9\Gamma^2 - 4)}\, t^{2-\frac{4}{3\Gamma}} - {}_{(2)}C^k{}_j \frac{2\Gamma}{a_f^2(\Gamma - 2)}\, t^{1-\frac{2}{\Gamma}} \right)\,, \tag{3.49}$$

$$\tilde{A}_{ij} = {}_{(2)}F_{ij}\, \frac{3\Gamma}{a_f^2(3\Gamma + 2)}\, t^{1-\frac{4}{3\Gamma}} + {}_{(2)}C_{ij}\, \frac{1}{a_f^3}\, t^{-\frac{2}{\Gamma}}, \tag{3.50}$$

$$\delta = {}_{(2)}Q\, t^{2-\frac{4}{3\Gamma}}\,, \tag{3.51}$$

$$u_i = {}_{(0)}L^{-6}\, \bar{D}^j \left[{}_{(0)}L^6\,{}_{(2)}F_{ij}\right] \frac{9\Gamma^2}{4a_f^2(3\Gamma + 2)}\, t^{3-\frac{4}{3\Gamma}} + {}_{(0)}L^{-6}\, \bar{D}^j \left[{}_{(0)}L^6\,{}_{(2)}C_{ij}\right] \frac{3\Gamma}{4a_f^3}\, t^{2-\frac{2}{\Gamma}}\,, \tag{3.52}$$

where freely specifiable functions are ${}_{(0)}L$, ${}_{(0)}f_{ij}$ and ${}_{(2)}C_{ij}$. The functions ${}_{(2)}F_{ij}$ and ${}_{(2)}Q$ are given by Eqs. (3.33) and (3.36), respectively, as functions of ${}_{(0)}L$ and ${}_{(0)}f_{ij}$. It may be noted that ${}_{(0)}L^4{}_{(2)}F_{ij}$ is the traceless part of $R_{ij}\left[L^4 f\right]$, i.e., the traceless part of the Ricci tensor of the metric ${}_{(0)}L^4{}_{(0)}f_{ij}$, as seen from its definition (3.33).

To clarify the physical role of these freely specifiable functions, let us first count the degrees of freedom. Since the determinant of ${}_{(0)}f_{ij}$ is unity, it has 5 degrees of freedom, and since ${}_{(2)}C_{ij}$ is traceless, it also has 5 degrees of freedom. So, together with the degree of freedom of ${}_{(0)}L$, the total number is $1 + 5 + 5 = 11$, while the true physical degrees of freedom are $4+2+2=8$, where 4 are of the fluid (1 from the density and 3 from the 3-velocity) and $2+2$ are of the gravitational waves (2 from the metric and 2 from the extrinsic curvature). This implies that there still remains 3 gauge degrees of freedom. It is easy to understand that these 3 gauge degrees comes from spatial covariance, that is, from the gauge freedom of purely spatial coordinate transformations $x^i \to \bar{x}^i = f^i(x^j)$. Thus we may regard that ${}_{(0)}f_{ij}$ contains these 3 gauge degrees of freedom.

To summarize, the degrees of freedom contained in the freely specifiable functions can be interpreted as

$$\begin{aligned} {}_{(0)}L &\quad \cdots \quad 1 = 1 \text{ (density)}\,, \\ {}_{(0)}f_{ij} &\quad \cdots \quad 5 = 3 \text{ (gauge) } + 2 \text{ (GWs)}\,, \\ {}_{(2)}C_{ij} &\quad \cdots \quad 5 = 3 \text{ (velocity) } + 2 \text{ (GWs)}\,. \end{aligned} \tag{3.53}$$

To fix the gauge completely, one therefore has to impose 3 spatial gauge conditions on ${}_{(0)}f_{ij}$. If ${}_{(0)}f_{ij}$ were not a metric, we could impose the transversality (divergence-free) condition on it. However, this cannot be done because ${}_{(0)}f_{ij}$ is the metric and any covariant derivative of it vanishes identically, the very nature of any metric tensor.

However, in the case of linear perturbation theory, or even in the case of a higher order perturbation theory, this difficulty disappears because of the presence of the background metric. In such a case, we may impose the transversality condition on the full (nonlinear) metric with respect to the background metric to single out the gravitational wave degrees of freedom. In fact, a set of most commonly used conditions is to impose the transversality with respect to the flat Cartesian metric,

$$\delta^{kj}\partial_k {}_{(0)}f_{ij} \equiv \partial^j {}_{(0)}f_{ij} = 0\,. \tag{3.54}$$

At this stage, it is useful to clarify the correspondence of the degrees of freedom counted in Eq. (3.53) to those of linear theory. From the time-dependence associated with ${}_{(0)}L$, ${}_{(0)}f_{ij}$ and ${}_{(2)}C_{ij}$, one can identify ${}_{(0)}L$ to the growing adiabatic mode of density perturbations, the 2 degrees in ${}_{(0)}f_{ij}$ to the growing modes of gravitational waves, the 3 degrees in ${}_{(2)}C_{ij}$ to 1 decaying scalar and 2 vector (vorticity) modes, and the remaining 2 in ${}_{(2)}C_{ij}$ to the decaying gravitational wave modes.

Now, returning to full nonlinear theory, it is generally impossible to identify uniquely the gravitational wave degrees of freedom in the metric because of general covariance. Nevertheless, if we focus on the extrinsic curvature K_{ij}, since its transverse-traceless part may be defined unambiguously, we may be able to single out the gravitational wave (tensor) degrees of freedom.

To show this is indeed the case, let us consider the momentum constraint (3.37),

$$\bar{D}^j\left[{}_{(0)}L^6\tilde{A}_{ij}\right] = 8\pi G\Gamma\rho_0 u_{i(0)}L^6 + O(\epsilon^5). \tag{3.55}$$

If we require the tensor modes to be transverse-traceless, an apparent candidate is the part of ${}_{(0)}L^6\tilde{A}_{ij}$ that does not contribute to the right-hand side of the momentum constraint. Namely, we identify the transverse-traceless (TT) part of ${}_{(0)}L^6\tilde{A}_{ij}$ with respect to the metric ${}_{(0)}f_{ij}$ as the tensor modes. Then the question is if the TT part can be uniquely determined. Fortunately, this is known to be possible [31]. Here we recapitulate the result briefly. For any symmetric traceless second rank tensor X_{ij} (with respect to $\tilde{\gamma}_{ij}$ in our case), we have the decomposition,

$$\begin{aligned} X_{ij} &= \tilde{D}_i W_j + \tilde{D}_i W_i - \frac{2}{3}\tilde{\gamma}_{ij} D^k W_k + X_{ij}^{TT} \\ &\equiv (\Lambda W)_{ij} + X_{ij}^{TT}\,, \end{aligned} \tag{3.56}$$

where W_i satisfies the second-order differential equation,

$$(O\,W)_i \equiv -\tilde{D}^j(\Lambda W)_{ij} = -\tilde{D}^j(X_{ij})\,. \tag{3.57}$$

York showed that the solution is uniquely determined [31]. He then further showed that this decomposition is conformally invariant. That is, for $\hat{\gamma}_{ij} = \phi^4\tilde{\gamma}_{ij}$, one has $\hat{X}_{ij} = \phi^{-2}X_{ij}$, and so as the TT part,

$$\hat{X}_{ij}^{TT} = \phi^{-2}X_{ij}^{TT}\,. \tag{3.58}$$

Thus, identifying X_{ij} with ${}_{(0)}L^6\tilde{A}_{ij}$, we immediately see that the transverse parts of ${}_{(0)}L^6\,{}_{(2)}F_{ij}$ and ${}_{(0)}L^6\,{}_{(2)}C_{ij}$ represent the gravitational wave modes. Furthermore, from their time-dependence, it is clear that the former is the growing mode while the latter is the decaying mode, as in the case of linear theory.

Finally, let us comment of the vector mode. It is contained only in ${}_{(2)}C_{ij}$, This can be seen by considering the vorticity conservation law which is valid for any perfect fluid [32]. One finds that the time-dependence $t^{2-\frac{2}{\Gamma}} \propto a^{3(\Gamma-1)}$ associated with ${}_{(2)}C_{ij}$ is precisely the one resulting from the vorticity conservation.

IV. COORDINATE TRANSFORMATION TO THE SYNCHRONOUS COORDINATES

Comer et al. [24] obtained the metric to $O(\epsilon^2)$ in the synchronous coordinates. However, the result they obtained, Eq. (2.14) of their paper, looks rather different from ours, partly because of the difference in the choice of coordinates and partly because of the fact that their Eq. (2.14) is not completely the general solution. Here, we look for a coordinate transformation from our coordinates to the synchronous coordinates and derive the explicit relation between their solution and ours.

Let us denote the coordinates in the uniform-Hubble slicing by $\{x^\mu\}$ and those in the synchronous gauge by $\{\bar{x}^\mu\}$, and the corresponding spacetime metrics by $g_{\mu\nu}(x)$ and $\bar{g}_{\mu\nu}(\bar{x})$, respectively. The solution obtained by Comer et al. is

$$\begin{aligned} \bar{g}_{ij} \;&=\; t^{\frac{4}{3\Gamma}}\Bigg[h_{ij}+\Big(-\frac{3\Gamma^2}{4(9\Gamma-4)}t^{2-\frac{4}{3\Gamma}}+\gamma_2' t^{-1}+\gamma_2\Big)h_{ij}R[h] \\ &\qquad +\Big(-\frac{9\Gamma^2}{9\Gamma^2-4}t^{2-\frac{4}{3\Gamma}}+\beta_2' t^{1-\frac{2}{\Gamma}}+\beta_2\Big)\Big(R_{ij}[h]-\frac{h_{ij}}{3}R[h]\Big)\Bigg], \end{aligned} \tag{4.1}$$

where

$$h_{ij} = {}_{(0)}L^4\,{}_{(0)}f_{ij}\,, \tag{4.2}$$

and $R_{ij}[h]$ and $R[h]$ are the Ricci tensor and Ricci scalar, respectively, of the metric h_{ij}, and β_2, β_2', γ_2 and γ_2' are arbitrary constants. Here and in the rest of this section, we set $a_f = 1$ in accordance with the convention of [24]. We note that the terms containing β_2 and γ_2 can be absorbed into h_{ij}, hence we may effectively set these constants to zero without loss of generality. Thus the metric we consider is

$$\begin{aligned} \bar{g}_{ij} \;&=\; t^{\frac{4}{3\Gamma}}\Bigg[h_{ij}+\left(-\frac{3\Gamma^2}{4(9\Gamma-4)}t^{2-\frac{4}{3\Gamma}}+\gamma_2' t^{-1}\right)h_{ij}R[h] \\ &\qquad +\left(-\frac{9\Gamma^2}{9\Gamma^2-4}t^{2-\frac{4}{3\Gamma}}+\beta_2' t^{1-\frac{2}{\Gamma}}\right)\left(R_{ij}[h]-\frac{h_{ij}}{3}R[h]\right)\Bigg]. \end{aligned} \tag{4.3}$$

It should be noted that the term $\gamma_2' t^{-1}$ in the first line of the above is a gauge mode that exists in the synchronous gauge, corresponding to a shift of the time slice $t \to t + \delta t(\bar{x}^i)$.

Now consider a coordinate transformation,

$$x^\mu \to \bar{x}^\mu = f^\mu(x)\,. \tag{4.4}$$

The metric is transformed as

$$g_{\mu\nu}(x) = \frac{\partial \bar{x}^\alpha}{\partial x^\mu}\frac{\partial \bar{x}^\beta}{\partial x^\nu}\bar{g}_{\mu\nu}(\bar{x})\,. \tag{4.5}$$

For $\bar{g}_{\mu\nu}$ being the metric in the synchronous coordinates, that is, for $\bar{g}_{00} = -1$ and $\bar{g}_{0i} = 0$, we have

$$\begin{aligned} g_{00}(x) \;&=\; -(1+2\chi)+O(\epsilon^4) = -\frac{\partial f^0}{\partial t}\frac{\partial f^0}{\partial t}+\frac{\partial f^i}{\partial t}\frac{\partial f^j}{\partial t}\bar{g}_{ij}(\bar{x})\,, \\ g_{k0}(x) \;&=\; \beta_k = O(\epsilon^3) = -\frac{\partial f^0}{\partial t}\frac{\partial f^0}{\partial x^k}+\frac{\partial f^i}{\partial t}\frac{\partial f^j}{\partial x^k}\bar{g}_{ij}(\bar{x})\,. \\ g_{kl}(x) \;&=\; -\frac{\partial f^0}{\partial x^k}\frac{\partial f^0}{\partial x^l}+\frac{\partial f^i}{\partial x^k}\frac{\partial f^j}{\partial x^l}\bar{g}_{ij}(\bar{x})\,. \end{aligned} \tag{4.6}$$

Since the two metrics should coincide with each other in the limit $\epsilon \to 0$, and the non-trivial corrections are $O(\epsilon^2)$ in both metrics, we may set the coordinate transformation to be of the form,

$$f^\mu(x) = x^\mu + F^\mu(x)\,, \tag{4.7}$$

where $F^\mu = O(\epsilon^2)$. Here we have assumed that the spatial coordinates also coincide in the limit $\epsilon \to 0$, although there always exist 3 purely spatial coordinate degrees of freedom as discussed around Eq. (3.54). Inserting the above to the first two of Eqs. (4.6), we readily find

$$\frac{\partial F^0}{\partial t} = \chi\,, \tag{4.8}$$

and

$$a^2 h_{ij}\frac{\partial F^j}{\partial t}=\frac{\partial F^0}{\partial x^i} \quad \leftrightarrow \quad \frac{\partial F^i}{\partial t}=a^{-2}h^{ij}\frac{\partial F^0}{\partial x^j}\,, \tag{4.9}$$

while the last of Eqs. (4.6) gives

$$\begin{aligned} g_{ij}(x) &= \bar{g}_{ij}(\bar{x})+O(\epsilon^3)=\bar{g}_{ij}(x)+F^0(x)\partial_t\bar{g}_{ij}(x)+O(\epsilon^3) \\ &= \bar{g}_{ij}(x)+2H(t)\,F^0(x)g_{ij}(x)+O(\epsilon^3)\,. \end{aligned} \tag{4.10}$$

Equations (4.8) and (4.9) determine the coordinate transformation. Integrating Eq. (4.8), we find

$$F^0=\int^t \chi\,dt=-\frac{3\Gamma-2}{9\Gamma-4}{}_{(2)}Q\,t^{3-\frac{4}{3\Gamma}}+{}_{(2)}E(x^i)=-\frac{3\Gamma^2(3\Gamma-2)}{8(9\Gamma-4)}\,R[h]\,t^{3-\frac{4}{3\Gamma}}+{}_{(2)}E(x^i)\,, \tag{4.11}$$

where ${}_{(2)}E(x^i)$ is a function of the spatial coordinates (to be determined below), and the second equality follows from Eq. (3.36). Substituting this into F^0 in Eq. (4.10) and noting Eq. (4.3), we find the metric in the uniform Hubble slicing as

$$\begin{aligned} g_{ij}(x) &= t^{\frac{4}{3\Gamma}}\Bigg[h_{ij}\left(1-\frac{\Gamma}{4}R[h]\,t^{2-\frac{4}{3\Gamma}}+\left\{\gamma_2' R[h]+\frac{2}{3\Gamma}{}_{(2)}E\right\}t^{-1}\right) \\ &\qquad +\left(-\frac{9\Gamma^2}{9\Gamma^2-4}t^{2-\frac{4}{3\Gamma}}+\beta_2' t^{1-\frac{2}{\Gamma}}\right)\left(R_{ij}[h]-\frac{h_{ij}}{3}R[h]\right)\Bigg]. \end{aligned} \tag{4.12}$$

Apparently, the terms in the curly brackets proportional to t^{-1} should vanish for the uniform Hubble slicing. This condition determines the function ${}_{(2)}E(x^i)$. In passing, it may be noted that the solution obtained by Comer et al. assumes the gauge mode to be proportional to $R[h]$, but it can be an arbitrary function of the spatial coordinates in general.

Now from Eqs. (3.48) and (3.49), and noting that $g_{ij}=t^{\frac{4}{3\Gamma}}\psi^4\tilde{\gamma}_{ij}$ and $h_{ij}={}_{(0)}L^4{}_{(0)}f_{ij}$, our solution may be expressed as

$$g_{ij}(x)=t^{\frac{4}{3\Gamma}}h_{ik}\left(1-\frac{\Gamma}{16}R[h]\,t^{2-\frac{4}{3\Gamma}}\right)^4\left(\delta^k_j-\frac{9\Gamma^2}{9\Gamma^2-4}{}_{(2)}F^k{}_j\,t^{2-\frac{4}{3\Gamma}}+{}_{(2)}C^k{}_j\,t^{1-\frac{2}{\Gamma}}\right), \tag{4.13}$$

where we have from Eq. (3.33),

$${}_{(2)}F^k{}_j=\left(R^k{}_j[h]-\frac{1}{3}\delta^k_j R[h]\right). \tag{4.14}$$

Comparing this with Eq. (4.12), we see that these two metrics are identical provided that we choose

$$\frac{2\Gamma}{\Gamma-2}{}_{(2)}C^k{}_j=\beta_2'\;{}_{(2)}F^k{}_j\,. \tag{4.15}$$

Thus we have successfully clarified the relation between our solution and the solution obtained by Comer et al. [24]. As clear from this equation, their solution corresponds to a solution with a special choice of ${}_{(2)}C_{ij}$, similar to the choice of their gauge mode mentioned before. Although the choice of ${}_{(2)}C_{ij}$ is rather unimportant in the present case of a perfect fluid because it contains only decaying modes, Such a special choice is not only required but also renders the physical meaning of ${}_{(2)}C_{ij}$ unclear.

V. CONCLUSION

In this paper, focusing on the simple case of a perfect fluid with constant adiabatic index, we have considered nonlinear perturbations on superhorizon scales in the context of (spatial) gradient expansion. We have adopted the uniform Hubble slicing and derived the general solution valid to second order in spatial gradients. The general solution is found to contain 11 arbitrary functions of the spatial coordinates. We have successfully identified the physical meaning of all these functions. In particular, the relation of these degrees of freedom to those in the linear perturbation theory has been clarified. This is summarized as follows.

$$\begin{aligned}
1 \quad &\cdots \quad \text{growing scalar (curvature) perturbation}\,, \\
1 \quad &\cdots \quad \text{decaying scalar perturbation}\,, \\
2 \quad &\cdots \quad \text{decaying vector (vorticity) perturbations}\,, \\
2 \quad &\cdots \quad \text{growing tensor perturbations}\,, \\
2 \quad &\cdots \quad \text{decaying tensor perturbations}\,, \\
3 \quad &\cdots \quad \text{spatial gauge degrees of freedom}\,.
\end{aligned} \tag{5.1}$$

In doing this identification of various modes, we have found that the tensor modes contained in the extrinsic curvature (the time derivative of the metric) can be identified only non-locally. The detailed understanding of this result and its implications are left for future studies.

Then we have compared our result with the one obtained by Comer et al. in the synchronous gauge [24] by finding the coordinate transformation from the uniform Hubble slicing to the synchronous (proper time) time-slicing. We have found that their solution corresponds to a special choice of $5 = 1 + 2 + 2$ decaying modes (i.e., 1 scalar, 2 vector and 2 tensor decaying modes) of the 11 arbitrary functions mentioned above.

Although these decaying modes are rather unimportant in the present case of a perfect fluid, they may not be negligible in the case of a scalar field. In particular, as mentioned in Introduction, the decaying modes may play a crucial role even in the case of a single component scalar field if the conventional slow-roll condition is violated. In this respect, one of the advantages of our framework presented in this paper is that it can take account of the whole physical degrees of freedom.

Finally, we note that our result contains not only nonlinear scalar modes but also the nonlinear coupling of scalar, vector and tensor modes. Thus, although the present paper is limited to the case of a simple perfect fluid, if the framework we have developed is extended to the case of a (multi-component) scalar field, the resulting formalism will be a powerful tool to investigate various non-linear, non-local effects from the inflationary cosmology, especially non-Gaussianity of curvature perturbations from inflation. We plan to come back to this issue in the near future.

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APPENDIX A: ORDER COUNTING

Here we demonstrate explicitly the order counting of the basic variables given in Eq. (3.4). It is essentially a recapitulation of the analysis given in [30].

In the $(3+1)$-decomposition, u^0 is represented as

$$u^0 \quad = [\alpha^2 - (\beta_k + v_k)(\beta^k + v^k)]^{-1/2}.$$

From the first two basic assumptions (3.1), i.e., $v^i = O(\epsilon)$, $\beta^i = O(\epsilon)$, we have

$$u^0 = \frac{1}{\alpha} + O(\epsilon^2). \tag{A1}$$

This equation and Eq. (2.8) yield

$$E = (\rho + P)(\alpha u^0)^2 - P = \rho + O(\epsilon^2). \tag{A2}$$

Then from the Hamiltonian constraint (2.15), we obtain

$$\begin{aligned}
H^2 \;&=\; \frac{8\pi G\rho}{3} + O(\epsilon^2), \\
&=\; \frac{8\pi G\rho_0(1+\delta)}{3} + O(\epsilon^2).
\end{aligned} \tag{A3}$$

Now, since H is uniform by definition on uniform Hubble slices, we have

$$\delta = O(\epsilon^2). \tag{A4}$$

In general, we have

$$\psi = O(1)\,. \tag{A5}$$

From Eq. (2.12), we have

$$\partial_t w + 6\frac{\partial_t \psi}{\psi} w + \frac{\partial_t \delta}{\Gamma(1+\delta)} w + 6\frac{\partial_k \psi}{\psi} w v^k + \frac{w\rho_0 \partial_k \delta}{\Gamma \rho} v^k + w\partial_k v^k + (\partial_k w) v^k = 0\,, \tag{A6}$$

where $w \equiv \alpha u^0$. Then Eq. (A1) gives $w = 1 + O(\epsilon^2)$, and Eq. (A6) gives

$$\partial_t \psi = O(\epsilon^2)\,. \tag{A7}$$

From Eq. (2.10), we see

$$S_{ij} - \frac{\gamma_{ij}}{3} S^k{}_k = O(\epsilon^2)\,. \tag{A8}$$

Therefore, from Eq. (2.20) we have

$$\partial_t \tilde{A}_{ij} = \alpha K \tilde{A}_{ij} + O(\epsilon^2). \tag{A9}$$

The uniform Hubble slice condition (2.23) gives

$$\chi = \alpha - 1 = O(\epsilon^2). \tag{A10}$$

Then, Eq. (A9) becomes

$$\partial_t \tilde{A}_{ij} = -3\frac{\dot{a}}{a}\,\tilde{A}_{ij} + O(\epsilon^2)\,. \tag{A11}$$

The lowest order homogeneous solution of this equation is $\tilde{A}_{ij} \propto a^{-3}$ which could be of $O(\epsilon)$. As discussed in the text, however, we assume

$$\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2). \tag{A12}$$

This assumption, together with Eq. (2.18), implies that the $O(\epsilon)$ part of $\tilde{A}_{ij}$ is absent. We therefore have

$$\tilde{A}_{ij} = O(\epsilon^2)\,. \tag{A13}$$

Finally, from Eq. (2.16), we have

$$u_j = u^0(v_i + \beta_i) = O(\epsilon^3). \tag{A14}$$

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